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FOREWORD ON BEHALF OF THE EDITORS.

By E. R. HEDRICK, University of Missouri.

Collegiate Mathematics.—The lifework of hundreds of men is the teaching of collegiate mathematics; hundreds of other men are preparing for this same lifework. Thousands of students are now working and will work under the guidance of these men in collegiate courses in mathematics.

Suggestions of moment arise in the minds of many, yet there exist few means for transmitting an idea in this field to others. Any organized effort toward the dissemination of methods that are new, or of facts that are interesting, in this field is totally lacking.

These remarks apply, of course, only to the United States. Abroad, in every country in which mathematics holds a place commensurate with its position here, definite journals exist whose purpose is almost wholly to foster these interests. The AMERICAN MATHEMATICAL MONTHLY, beginning with this issue, proposes to afford an opportunity for any discussions that seem valuable upon collegiate mathematics, and the editors invite contributions concerning the methods of instruction as well as those that treat special topics or theorems.

Problems of Instruction.—Every teacher in this field has asked himself what topics he should best teach, why these topics are taught, how they should be approached, what purpose should be recognized as the main aim of collegiate mathematics. Ordinarily he has either found no answer whatever and has settled down to the basis of traditional procedure, or else he has satisfied himself perforce with partial answers that he could himself discover. Each man has been a power unto himself, but no man has received or given aid, at least to the extent of his ability.

Some conspicuous exceptions to this general state of affairs are mentioned below. What has been done is, however, sporadic, discontinuous, and of little influence,—certainly no such effort has been general, and the effects of what has been attempted are far from general.

The most general and most sustained work in this direction has recently culminated in the report of a committee of twenty mathematicians and engineers, under the chairmanship of Professor E. V. Huntington, of Harvard University.

This committee, was originally appointed at a joint meeting* of mathematicians and engineers, organized in 1907 by the Chicago Section of the American Mathematical Society; but it was authorized to report to the Society for the Promotion of Engineering Education, which body had already carried on a consistent series of discussions through a period of years on the teaching of mathematics to students of engineering; and these engineers, rather than the mathematicians, have furnished the medium, through their journal, for the publication of these discussions and of this final report.†

Though it may be contended that the aims of this Society are at least special, and touch only one side of our work, and though the spirit of many of the discussions may seem narrow to many, it is at least true that the attempt to suggest means for the improvement of mathematical teaching has been carried on consistently, courageously, and earnestly.

The efforts of this society, whose special aim is the improvement of engineering education, emphasize the variety of conditions that must be met by collegiate courses in mathematics at this time. The reasonable demands of the rapidly increasing schools of engineering, together with the normally increasing enrollments in academic classes, have vastly increased the problems that confront us.

Origin of New Problems.—The problems of to-day differ from those of a generation ago on account of three great movements that lie wholly outside the field of mathematics. Of these, the first is the spread and almost universal acceptance of the elective principle in colleges and in academic classes in institutions of every kind. To be sure, the tendency to-day seems to be to limit the scope of the elective system, but the days of the old fixed curriculum, with mathematics as a required study for all students, has passed, and forever.

Another movement of no less extent is the rapid increase in the number of those entering engineering and other semi-technical professions. In schools of engineering, a certain amount of mathematics is required of all students, though the amount varies. But this requirement that all students take collegiate courses in mathematics is not made, as was the older academic requirement, for the sake of general mental discipline and the training of the mind as a whole. While this motive may persist in part, the recent discussions by engineers, mentioned above, leave no room for doubt that the inherent reason, in this instance, is the belief in the direct usefulness to the engineer of the facts, the principles, and the processes to be learned in these courses.

The last, in point of time, of the three principal movements mentioned above is the present rapid change in educational theory, which is based to a large extent on experimental psychology, and which bears the same relation to the older pedagogy that chemistry bears to alchemy. Not all its problems are yet settled,

* A few printed copies of the papers read at this meeting may still be had from the secretary of the Chicago Section, H. E. Slaught, 58th Street and Ellis Ave., Chicago, Ill.

† This final report is in the form of a syllabus of courses in mathematics for students of engineering, and is for sale by the Society for the Promotion of Engineering Education, 43 E. Nineteenth St., New York.

many of them are now under intense discussion, but the manner of that discussion is the manner of science rather than the manner of controversy. One of its contributions that is widely accepted is that there are no such *fixed* faculties of the mind as those we had believed in rather implicitly; and that the training of such supposed faculties as the memory, for example, is impossible in its widest sense because there seems to be no one such faculty. The memory for number, for example, seems to be wholly unrelated to the memory for poetry, or the memory for forms.

Discipline.—To what extent training in any field carries over into any other field seems to be still under discussion. Whether the memory, for example, extends at all beyond the very region of the training received, whether a child who has been drilled in one special field gains in ability in another even closely allied field, seems yet not precisely known. That upon which there is complete agreement is that training of such a supposed faculty as the memory in *one* field does not strengthen the memory in *all* fields, but that it does strengthen the memory in at least some fields closely allied to that in which the training was received. The uncertainty that remains is to define how close the alliance must be.

While the question of training the memory is itself at least a minor consideration in the minds of many teachers of mathematics, the very analogous question of disciplinary training of the mind has been and is one of the strongest motives in the minds of many for the teaching of any course in mathematics. It was certainly the basis for the older requirement of mathematics in the early colleges. In so far as the modern theory of education renders uncertain the possibility of disciplinary training, it vitally affects every discussion of any mathematical topic. In how far training of the reasoning faculty may aid the student to better reasoning in general, in how far power to think in a particular field may be generally efficacious, are, in the opinion of many, problems whose solution lies in the future. We only know that the training received in reasoning, or thinking, upon one topic affects most strongly the ability to reason, or to think, on topics most closely associated with the topics in which the training was received. The uncertainty that is recognized is in describing accurately how close to the original topic a new one must be in order to show the effect of the previous training.

Selection of Topics.—Teachers of mathematics will not lightly abandon the value of the disciplinary training afforded by the subject, but the ability and scientific attitude of mind of teachers of collegiate mathematics, as compared with teachers in more elementary schools, will lead them to search for the truth about these questions and will make them cautious in basing their faith entirely upon an aim that is made questionable by these modern investigations in education.

One immediate result is the realization that the same amount of mental training, whatever that may be, seems likely to result from any one of several possible topics, if each of them is studied with the same thoroughness and atten-

tion. Does this make possible the selection of topics on grounds entirely different from mental discipline and, if the topics to be taught are selected for other reasons, will the resultant training be just as efficient?

This consideration, and the changed conditions which seem to raise the same demand, would point toward a very thorough and critical study of the motives, other than disciplinary training, for the study of mathematical courses, and of topics within those courses. Without discarding discipline as one of the desirable results of a mathematical training, a knowledge of other motives for the study of any one topic will serve only to enhance its value.

Comparative merits of different topics and of different methods of instruction can be realized most effectively if these other motives are considered, and if the disciplinary factors are temporarily ignored, for the claims of disciplinary value, as between two topics, are at best conflicting and uncertain, and the wholesome effect of any topic that is studied intensively will be recognized by all.

Interest.—The qualification that a study of any topic, in order to be effective, must be intensive and earnest will be insisted upon by all. It would seem therefore that profitable discussion may well take place concerning the means by which such work can be obtained. To arouse the student to his maximum effort, to stimulate his greatest efficiency, to gain the fullest measure of disciplinary good, there may be many plans of merit; but it is certain that all of them will have in common the holding of the student's attention, without which no effective study is thinkable. Any process for holding the attention in a sustained and serious fashion is described by saying that the student has become "interested" in the study or that his "interest" has been secured. But the word interest has been misused. The doctrine of interest has led to a mistaken doctrine of "amusement," than which no interpretation of the doctrine of interest could be more absurdly irrelevant. It is scarcely to be wondered at that the whole doctrine, sound as its real basis may be, has been misunderstood, abused, ridiculed, and that it appears to many as the antithesis of that very doctrine of discipline of which it is the sole effective support.

The Province of the Monthly.—This article is not itself a pedagogical document. What is here stated is perfectly well known in pedagogical circles, and we trust that we have not through ignorance, as we have certainly not through intention, overstepped the bounds of that which is generally recognized as finally established in the theory of education.

It is not the province of the MONTHLY to enter the field of general pedagogy, nor will the MONTHLY entertain discussions that are concerned with research in general pedagogy, or that deal with new theories of pedagogy, however important these contributions may seem. Journals for the publication of such contributions abound, and any article of worth in the general field is already more than liberally provided with possible means of publicity. In such matters, we should not venture to pose as expert critics of new matter, nor to pass upon the merits of articles. We must accept, as from a foreign field, the results of general pedagogy and of experimental psychology, and in order to be secure, we

must remain always a little behind, rather than in advance of the times, on pedagogical theory.

What we do desire is to inspire, not a discussion along these lines, but rather a discussion of definite *mathematical* problems. It is our hope that this article will spur mathematicians to the realization of the new conditions that confront us, to the variety of problems that demand discussion, to the vital importance which their solution holds to a great body of intelligent people. For the discussion from this standpoint of mathematical questions, properly speaking, the MONTHLY will open its columns without reserve, except for those usual discretionary and advisory powers that are traditional in any editorship.

The Future.—This is not the last word that the MONTHLY will have to say upon this question. We shall indulge in rather little general pedagogical preaching, for we claim no professional attitude toward that subject. What seemed necessary has been said above. For this issue this is enough. We shall, however, not place any limitation upon the scope of our suggestions to readers and to possible contributors; and we shall hope, by these suggestions, to point out very explicit problems of a mathematical nature which might well be treated. We shall aim to awaken an interest in their discussion, even among those who now regard their own profession—the teaching of collegiate mathematics—with distrust as a possible field for that type of human thinking that is known as scientific research. In this distrust, we, the editors of the MONTHLY, emphatically announce that we do not share.

HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

By FLORIAN CAJORI, Colorado College, Colorado Springs.

I. FROM NAPIER TO LEIBNIZ AND JOHN BERNOULLI I. 1614–1712.

LOGARITHMS OF POSITIVE NUMBERS.

The seventeenth and eighteenth centuries are notable in the history of algebra for important developments of the notations and ideas which mark its modern treatment. The study of this period in algebra is more than mere pastime. In the words of Felix Klein: "If we really desire to advance to a full understanding of the theory of logarithms, it is best to follow in broad outline the history of its creation."[1]

Logarithms were invented before our modern exponential notation, a^n, was introduced into algebra. To be sure, the use of algebraic symbols that were more or less different from the modern symbols to indicate powers and roots of a number had been suggested before the advent of the logarithm, but we shall

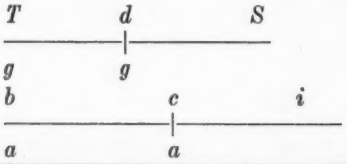
[1] F. Klein, *Elementarmathematik vom höh. Standpunkte aus*, I, Leipzig, 1908, p. 325.

see that these suggestions had remained unheeded; the fact is that the inventors of logarithms did not use the modern exponential notation and were not familiar with the exponential concept which now plays such a fundamental rôle in the development of logarithmic theory. What then were the basic considerations in the development of logarithms as entertained by their inventors, John Napier and Joost Bürgi?

John Napier's *Mirifici logarithmorum canonis descriptio*[1] appeared in 1614 in Edinburgh, and his *Mirifici logarithmorum canonis constructio* appeared there as a posthumous work in 1619, though written as early as, or earlier than, the *Descriptio*. Napier based his explanations upon two considerations: (1) The geometrico-mechanical concept of flowing points, (2) the relations which exist between arithmetic and geometric series. Several writers before the time of Napier called attention to certain relations between the terms of a geometric series and the terms of an arithmetic series, which relations involve the logarithmic idea. But those writers did not realize the possibilities of this idea nor did they conceive and execute the plan of computing a pair of corresponding series sufficiently dense for practical use in computation. From certain passages in authors like Stifel[2] one might be tempted to say that the logarithmic concept really existed before the time of Napier and Bürgi. Yet how much of a novelty the logarithms of Napier really were to the foremost mathematicians of his day can be realized by the enthusiasm with which men like Briggs and Kepler took up the new topic. So new and original did logarithms seem to Briggs that he left London for Scotland, to visit Napier, the inventor. Briggs addressed him thus:

"My Lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy viz., the logarithms."

Napier lets the point g move along the definite line TS with a diminishing velocity such that its velocity at T is to that at d, as the distance TS is to the distance dS. At the same time Napier lets a point a move along the line bc (which is of indefinite length) with a uniform velocity which is the same as the initial velocity of the point g. If the two points start to move at the same moment, and if g is at d when a is at c, then the length bc is defined as the logarithm of dS. Napier constructed tables for trigonometric computation. With that end in view he lets TS stand for the *radius*, assigning to it the value 10^7, while dS stands for a given *sine*. At that time trigonometric functions were not thought of strictly as *ratios*. In the language of Napier, the definition of a logarithm is here as follows:



[1] A reprint of the entire *Descriptio* is found in Maseres' *Scriptores logarithmici*, Vol. 6, London, 1807, pp. 475–624.

[2] Tropfke, *Gesch. d. Elementar-Mathematik*, Vol. II, 1903, pp. 141–145.

"The logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which radius began to decrease geometrically, and in the same time as radius has decreased to the given sine."[1]

Letting $v = 10^7$, the geometric and arithmetic series of Napier may be exhibited in modern notation as follows:

$$\begin{array}{llllll} v, & v(1-1/v), & v(1-1/v)^2, & \cdots & v(1-1/v)^v & \cdots \\ 0, & 1, & 2, & \cdots & v & \cdots \end{array}$$

The numbers in the upper series represent successive values of the *sines;* the numbers in the lower series stand for the corresponding logarithms. Thus, $\log 10^7 = 0$, $\log (10^7 - 1) = 1$, and generally, $\log (10^7\{1 - 10^{-7}\})^n = n$, where $n = 0, 1, 2, \cdots$. Napier had a definite object in view in making the logarithm of the radius, 10^7, equal to zero. He looked upon this arrangement as labor-saving, considering the frequency with which multiplications and divisions by the radius arise in trigonometric computation. He made the geometric series decrease while the corresponding arithmetic series increases. This makes the logarithms of the sines of all angles between 0° and 90° positive; in our modern tables these logarithms are negative.

In the *Descriptio* logarithms are defined as follows: *Logarithmi sunt numeri qui proportionalibus adjuncti aequales servant differentias.*" (Logarithms are numbers which correspond to proportional numbers and have equal differences.) The proportional numbers are the terms of the geometric progression; the numbers having equal differences are the terms of the arithmetic progression. The word "logarithm" is of Greek structure and signifies "number of ratios." The idea is this: $v(1 - 1/v)^n$ is gotten from v by n successive applications of the ratio $(1 - 1/v)$. Hence n, which is the logarithm, indicates "the number of ratios." Napier restates the definition in 1614 as follows: *Logarithmi dici possunt numerorum proportionalium comites æquidifferentes.* (Logarithms may be called equidifferent companions to proportional numbers.)

Not only is Napier's definition of a logarithm different from the modern definitions, but the notion of a "base" is inapplicable to his system. To force the concept of a "base" upon his system we must modify it somewhat. If each number of the two progressions of Napier is divided by 10^7, so that 0 becomes the logarithm of 1, then 1 is the logarithm of $(1 - 1/10^7)^{10^7}$, which is nearly equal to e^{-1}, where e is the base of the natural system. Hence the base of Napier's logarithms, when modified as here indicated, is very nearly e^{-1}.

It is well-known that Joost Bürgi invented logarithms independently of Napier, but he lost all rights of priority by failure to publish until the praises of Napier's book began to resound throughout Europe. In 1620 appeared in Prag the *Progress-Tabulen,* containing Bürgi's logarithmic tables, but omitting

[1] *The Construction of the wonderful Canon of Logarithms* by JOHN NAPIER. Translated from Latin into English by W. R. MACDONALD, Edinburgh and London, 1889, p. 19.

the explanations of them that were promised on the title-page.[1] Hence his logarithms were unintelligible to the ordinary reader. Common to Bürgi and Napier was the use of progressions in defining logarithms. In Bürgi's tables the numbers in the arithmetic progression were printed in *red*, the numbers in the geometric progression were in *black*. The relation between Bürgi's logarithms, $10n$, and their antilogarithms is expressed in modern notation by the equation $10n = \log [10^8(1 + 1/10^4)^n]$, $n = 1, 2, 3, \cdots$. The notion of a "base" can no more be forced upon Bürgi's logarithms than it can be upon the logarithms in Napier's tables.[2] In neither system is $\log 1 = 0$. Their logarithmic concepts were more general than those of the present day in this respect, that by sliding one progression past the other they could select any positive number at random as the one whose logarithm is zero. We have seen that Napier originally chose $\log 10^7 = 0$ while Bürgi chose $\log 10^8 = 0$. The logarithms in their tables were integral numbers. More than this, the terms of the two series could be made to increase in the same direction or in opposite directions, at pleasure. That is, if $m > n$, one can make $\log m < \log n$, or $\log m > \log n$, just as one may choose. Napier originally chose the first alternative, Bürgi the second.

It is generally known that Napier and Briggs conferred with each other and agreed to modify the original logarithms of Napier. In the *Appendix* to Napier's posthumous work, the *Constructio*, an improvement is suggested, "which adopts a cypher as the Logarithm of unity, and 10,000,000,000 as the Logarithm of either one tenth of unity or ten times unity." The subsequent use of decimal fractions in logarithmic tables led to the common logarithm proper, in which $\log 1 = 0$ and $\log 10 = 1$. A readjustment of Napier's original logarithms was made in John Speidell's *New Logarithmes*,[3] published in 1619 in London, whereby the logarithms virtually became the so-called "natural logarithms" of to-day.

Since at that time logarithms were studied in connection with some geometric progression having a positive initial term and a positive ratio between successive terms, so that negative terms could not arise in the progression, the question did not force itself for consideration, what the logarithm of a negative number should be. So far as we have noticed, no writer of the seventeenth century has raised this question. We shall see that Leibniz raised it in 1712. It would not be surprising had it arisen earlier. Charles Reyneau in his *Analyse démontrée*, Paris, 1708, Vol. II, p. 802, gives the formula for differentiation, "$dl - x = -1dx/x$," but changes it in the table of errata to "$d - lx = -1dx/x$." Did Reyneau fail to receive the suggestion from the printers' devil's "$l - x$"? In the seventeenth century the theory of logarithms was really on a very satisfactory

[1] These explanations were printed for the first time in *Grunert's Archiv*, Vol. 26, 1856, p. 323, in an article giving valuable historical notes relating to Bürgi.

[2] Misleading is the statement of M. Koppe: ". . . die Byrgschen [logarithmen] haben zur Basis $(1 + 10^{-4})^{10000}$, d. h. eine rationale Zahl, die mit e auf 4 Stellen übereinstimmt" (*Bibliotheca mathematica*, 3d S., Vol. 3, 1902, p. 151), as well as the statement of Felix Klein: ". . . dass die Bürgische Basis $(1{,}0001)^{10000} = 2{,}718146 \ldots$ mit e bereits auf 3 Dezimalen übereinstimmt" (F. Klein, *Elementarmathematik von höh. Standpunkte aus*, I, Leipzig, 1908, p. 333). From Bürgi's tables we copy $\log 134983856 = 30000$. It is readily seen that the base $e = 2.718 \ldots$ does not fit here.

[3] John Speidell's book is reprinted in Maseres' *Scriptores logarithmici*, Vol. 6, 1807, pp. 728–759.

basis. The definition of logarithms was restricted so as to apply only to positive numbers; to every positive number there corresponded one and only one logarithm. The needs of the practical computer were met. No necessity arose then to extend the logarithmic concept to negative or complex numbers.

A revision of Napier's definition of the logarithm of positive numbers was made by Edmund Halley. In 1695[1] he used these words: "The Old definition of Logarithms, that they are *Numerorum proportionalium æquidifferentes comites*, is too fancy to define them fully: For they may much more properly be said to be *Numeri Rationum Exponentes*." "Thus, if there be supposed between 1 and 10 an infinite Scale of mean Proportionals, whose Number is 100,000, etc., *in infinitum;* between 1 and 2 there shall be 30102, etc., of such Proportionals, and between 1 and 3 there will be 47712, etc., of them, which Numbers therefore are the Logarithms of the Rationes, of 1 to 10, 1 to 2, and 1 to 3; and not so properly to be called the Logarithms of 10, 2, and 3." Halley's article, from which we quote, is obscurely and inaccurately worded, but possesses the merit of giving the first derivation, by processes divorced from geometry (the hyperbola), of infinite series for the computation of logarithms.

Halley's idea of logarithms of "ratios" involves an interesting point which historians have hitherto completely overlooked. Halley anticipated Roger Cotes by nineteen years in the introduction of a certain mode of measuring a ratio. Halley considers ratio a "*quantitas sui generis*." Yet Halley's idea finds its inception in Napier's ratios of distances of two flowing points. Halley says: "These Rationes we suppose to be measured by the Number of Ratiunculae contained in each." This number stands for the logarithm of the ratio. But "logarithms thus produced may be of as many forms as you please." If in place of 100,000 mean proportionals between 1 and 10 we take 230258 of them, then, instead of common logarithms, we get the natural logarithms. Every system of logarithms differs by a constant factor from some system chosen as a standard. Hence, for a given ratio, the number of ratiunculæ is arbitrary; it is equal to the number in the standard system, multiplied by a constant. In other words, *the measure of a ratio is a constant times the standard logarithm of that ratio.* This is the idea, here expressed by Halley, later elaborated by Cotes, then largely lost sight of until the nineteenth century, when F. Klein introduced it anew, in order that he might establish the validity of the processes of projective geometry for the hypotheses of non-euclidean geometry.[2]

The following quotation from the article "Logarithms" in the second edition (1743) of E. Stone's *New Mathematical Dictionary* is of interest, because it asserts the dependence of Cotes upon Halley and especially because it gives utterance to a feeling of strangeness toward Halley's definition of the "measure of a ratio"

[1] *Philosophical Transactions*, London, No. 216, p. 58.

[2] Klein, *Götting. Nachrichten*, 1871, No. 17, and *Mathem. Annalen*, Vol. 4, 1871, pp. 573–625. It is possible that even before Halley this idea of the measure of a ratio may have been entertained by Kepler, for he speaks of "*proportio, ejusque mensura, logarithmus.*" See Kepler, *Tabulæ Rudolphinæ*, Ulm, 1627, Cap. III, p. 11. In fact, Charles Hutton, in his *Philosophical and Mathematical Dictionary*, London, 1815, Art. "Exponent," mentions "the idea of measures of ratios, as delivered by Kepler, Mercator, Halley, Cotes, etc."

which seizes one who has not gone beyond the elementary concepts of measurement. Stone says:

"Mr. Cotes too, at the Beginning of his *Harmon. Mensur.* has done this Business in imitation of Dr. Halley, altho' more short, yet with the same Obscurity: for I appeal to any one, even of his greatest Admirers, if they know what he would be at in his first Problem, *viz. to find the Measure of a Ratio* from the Terms of the Problem itself. . . ."

"Logarithms the measures of ratios" is referred to by Saunderson in his *Algebra*,[1] and by W. J. G. Karsten of Halle, in an important paper on logarithms,[2] and by J. F. Lorenz, in his *Elemente der Mathematik*,[3] but in general, this definition failed to influence mathematical thought of the eighteenth century.

The early explanations of the use of logarithms involved the consideration of many special cases. The theorems $\log a + \log b = \log ab$, $\log a - \log b = \log a/b$, $\log a^m = m \log a$, were explicitly stated by Oughtred (not in algebraic symbols, as here, but in words) in his booklet *De æquationum affectarum resolutione in numeris*, which was published in 1652, bound in one volume with his *Clavis mathematica*. The 1631 edition of the *Clavis* does not give these theorems.[4]

The theoretic viewpoint of the logarithm was broadened somewhat during the seventeenth century by the graphic representation, both in rectangular and polar coördinates, of a variable number and its variable logarithm. Thus were invented the *logarithmic curve* and the *logarithmic spiral*. The graphic representation of functions was then just beginning to be understood. Probably the logarithmic curve suggested itself to many minds. It has not been possible thus far to ascertain with certainty who was the first inventor of it. Hutton says[5] that the curve has been attributed to Edmund Gunter, but Hutton could not find it in Gunter's writings. Most likely the rumor is due to a confusion between the "logarithmic curve," and the "logarithmic line" invented by Gunter and known as the forerunner of the slide rule. It has been thought that the earliest reference to the logarithmic curve was made by the Italian Evangelista Torricelli in a letter of the year 1644,[6] but Paul Tannery has made it practically certain that Descartes knew the curve in 1639.[7]

It is found in a work of J. Gregory of the year 1667[8] and is clearly explained in the second edition (1690) of a book by the French mathematician, C. F. M. Dechales. In the same year Christiaan Huygens made known without proof the beautiful properties of the logarithmic curve;[9] these properties were proved later

[1] *Select Parts of Professor Saunderson's Elements of Algebra*, 3d ed., London, 1771, p. 402.

[2] *Abh. Münch. Acad.*, V, 1768.

[3] J. F. Lorenz, *Elemente*, I Theil, Leipzig, 1793, p. 140.

[4] H. Bosmans, "La première édition de la Clavis mathematica d'Oughtred," *Annales de la Société scient. de Bruxelles*, T. 35, 2. partie, p. 38.

[5] C. Hutton, *Math. Tables*, London, 1811, Introduction, p. 84.

[6] G. Loria in *Bibliotheca mathematica*, 3d S., Vol. I, 1900, p. 75.

[7] *L'intermédiaire des mathématiciens*, T. VII, 1900, p. 95. Tannery refers to a letter of Descartes, dated Feb. 20, 1639 (Édit. of Descartes, 1677, Vol. III, letter 71).

[8] J. Gregory, *Geometriæ pars universalis*, Venetiæ, 1667; See Montucla, *Histoire des mathématiques*, II, 1799, p. 85.

[9] C. Huygens, *De la cause de la pesanteur*, published in 1690 as an appendix to the *Traité de la lumière*.

by G. Grandi[1] and G. Fontana.[2] Theorems on the quadrature of this curve were given by Torricelli, Huygens and J. Craig.[3] This curve was discussed also by J. Bernoulli.[4] Its rectification was first explained in 1692 by Marquis l'Hospital in a letter to Leibniz.[5] We shall see that during the eighteenth century the logarithmic curve plays a leading rôle in the discussion of the theory of logarithms, and that it helped to cloud rather than clarify the questions at issue.

The curve which is the graphic representation in polar coordinates of the relation between a variable and its logarithm was invented by René Descartes. It is described by him in 1638 in a letter to P. Mersenne.[6] Descartes does not give its equation, nor does he connect it with logarithms. He describes it as the curve which makes equal angles with all the radii drawn through the origin. Soon after, this spiral was re-invented by Torricelli, who, as we have seen, is second only to Descartes in being unquestionably associated with the history of the logarithmic curve. The name, *logarithmic spiral*, was coined by Pierre Varignon in a paper presented to the Paris academy in 1704 and published in 1722.[7] In England this spiral commanded the attention of Oughtred, Collins, Wallis and Barrow.[8]

No less important in the history of logarithms is a third curve, namely the hyperbola. The quadrature of the space between the hyperbola and its asymptotes was effected by Gregory St. Vincent in Book VII of his *Opus geometricum*, Antwerp, 1647.[9]

This area, for the rectangular hyperbola $xy = 1$, is expressed now in the logarithmic form $\log y_1/y$, the area thus indicated being bounded by the axis of x, the ordinates y and y_1, and the hyperbolic arc. Nevertheless, this investigation of Gregory St. Vincent, strictly speaking, does not figure in the history of logarithms. He does not mention logarithms. His result is purely geometric and would remain unaltered in every particular, had logarithms never been invented. What he established was simply the theorem that, if parallels to one asymptote are drawn between the hyperbola and the other asymptote, so that the successive areas of the mixtilinear quadrilaterals thus formed are equal, then the lengths of these parallels form a geometric progression.[10]

Apparently the first writer to state this theorem in the language of logarithms

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[1] *Geometrica demonstratio theorematum Hugeniorum circa logisticam seu logarithmicam*, Florenz, 1701.

[2] *Sopra il centro di gravità della logistica finita ed infinitamente lunga*, Torino, Mem. X and XI. See also Loria, *Ebene Kurven*, Deutsche Ausgabe von F. Schütte, Leipzig, 1902, p. 543.

[3] *The quadrature of the logarithmic curve* in *Philosoph. Transactions*, No. 242, 1698.

[4] *Acta Eruditorum*, 1696, p. 216.

[5] Leibniz, *Werke*, Ed. Gerhardt, Vol. II, p. 216.

[6] *Œuvres de Descartes*, éd. Cousin, VII, Paris, 1824, pp. 336–37; Éd. Adam et Tannery, II, Paris, 1898, p. 360. See also G. Loria, *Ebene Kurven*, Leipzig, 1902, pp. 448–456.

[7] Loria, *op. cit.*, p. 444.

[8] A. Favaro, *Bibliotheca Mathematica*, N. F., 5, 1891, pp. 23–25.

[9] Karl Bopp, "Die Kegelschnitte d. Gregorius a St. Vincentio," Leipzig, 1907, in *Abhandl. z. Gesch. d. math. Wiss.* (M. Cantor), XX Heft.

[10] K. Bopp, *op. cit.*, p. 265, Propos. CXXX in book on Hyperbola. A different mode of statement is given in Propos. CXXV.

was the Belgian Jesuit Alfons Anton de Sarasa,[1] who defended Gregory St. Vincent against attacks made by Mersenne. This statement was a very natural step to take, in view of Gregory St. Vincent's theorem that, in a hyperbola $xy = 1$, the asymptotic area varies in arithmetic ratio when y or x varies in geometric ratio, and of Napier's definition of a logarithm which established a one-to-one correspondence between a geometric series and an arithmetic series.

An important publication was Nicolaus Mercator's booklet, the *Logarithmotechnia*, London, 1668. Mercator writes the equation of the hyperbola in the form $y = 1/(1 + a)$, where $1 + a$ is the abscissa, and y the ordinate. He expands $1/(1 + a)$ by division into an infinite series (in itself a novel undertaking),

$$\frac{1}{1+a} = 1 - a + aa - a^3 + \cdots.$$

He gives a crude explanation of the process of summation which we now indicate by $\int x^n dx = x^{n+1}/(n+1)$, and then integrates the terms of the above series. One would expect him to write down the logarithmic series $\log(1 + a) = a - aa/2 + a^3/3 - \cdots$, which is attributed to him, but he does not do so. Instead of this he writes down the numerical values of the first few terms of that series, taking $a = .1$, thereby obtaining the area of the mixtilinear quadrilateral between the ordinates $y = 1$ and $y = 1 \div 1.1$ as .095310181. He repeats this process for $a = .21$.[2] Probably using the results reached by Gregory St. Vincent and de Sarasa, Mercator finally connects his own results with logarithms. That he really used the logarithmic series is evident also from the next step, which must have been obtained by integration of the terms of this series. He does not write down the general result of this integration, but again gives only the numerical values corresponding to the first few terms of the new series, for $a = .1$. Thus he obtains the value of $\int \log(1 + a)da$, for $a = .1$. These results are found at the very end of Mercator's booklet, under the caption, *Invenire summam logarithmorum*. Tropfke is of the opinion that Mercator's failure to print the general logarithmic series was due to a practice still prevalent in his day of merely hinting at new results, in order to maintain an advantage over others.[3] We do not consider this the real motive. Any reader could repeat Mercator's areal computations from the full explanations given. His course is probably due to a different conception of the proper style of exposition, which favored the concrete special case to the general formula. Wallis was the first to state Mercator's logarithmic series in general symbols.[4] While the investigations due to Gregory St. Vincent, Mercator and some others led to greatly improved methods of computing logarithms by infinite series, and to the phrasing of geometric theorems

[1] *Solutio Problematis a R. P. Marino Mersenno propositi*, 1649. See Cantor, *op. cit.*, Vol. II, 2d ed., pp. 714, 715.

[2] See Maseres, *Scriptores logarithmici*, Vol. I, London, 1791, p. 194. Maseres reprints here the entire *Logarithmo-technia*.

[3] Tropfke, *op. cit.*, II, 1903, p. 182.

[4] Maseres, *op. cit.*, Vol. I, p. 229; *Phil. Trans.*, No. 38, year 1668.

in the language of logarithms, no modification of the logarithmic concept resulted from these researches.

In a MS. of July, 1676, Leibniz[1] derives the integral $\int dy/y$. It arose in the study of the problem which Florimond de Beaune had propounded to Descartes who replied in a letter of Feb. 20, 1639.[2] To find the quadrature of that curve in which the ordinate is to the subtangent as a given line segment is to the difference of the ordinate and abscissa. In this MS. of 1676, as well as in the published paper, the *Nova methodus*,[3] Leibniz is led by this problem to the equation $w/a = dw/dx$. He takes dx as a constant b, and gets $w = a/b \cdot dw$; that is, the ordinates w are proportional to their increments. If the x's increase in arithmetic progression, then the corresponding w's are the terms of a geometric progression. Hence the x's are the logarithms of the w's; he concluded that the curve is logarithmic.[4] As early as 1675 Leibniz used the notation "Log y," as in the equation "$a^2 - yx = 2y^2$ Log y."[5]

The Modern Exponential Notation.

The modern symbolism for powers of numbers was introduced by René Descartes in his *La géométrie*, Paris, 1637. He writes "*aa* ou a^2 pour multiplier a par soimême; et a^3 pour le multiplier encore une fois par a, et ainsi à l'infini."[6] Thus, while Vieta represented A^3 by "A cubus" and Stevin x^3 by a figure 3 within a small circle, Descartes wrote a^3. In his *Géométrie* he does not use negative and fractional exponents, nor literal exponents. His notation was the outgrowth and an improvement of notations employed before him by Chuquet, Bombelli, Bürgi, Reymer, and Kepler. Chuquet's manuscript work, *Le Triparty en la science des nombres*, 1484, gives $12x^3$ and $10x^5$, and their product $120x^8$, by the symbols 12^3, 10^5, 120^8, respectively. Chuquet goes even further and writes[7] $12x^0$ and $7x^{-1}$ thus 12^0, 7^{1m}. He represents the product of $8x^3$ and $7x^{-1}$ by 56^2. Bürgi, Reymer and Kepler use Roman numerals for the exponential symbol. Bürgi writes $16x^2$ thus 16^{II}. Thomas Harriot simply repeats the letters; he writes[8] $a^4 - 1024a^2 + 6254a$ thus: *aaaa* − 1024*aa* + 6254*a*.

Descartes's notation spread rapidly. It was used by Fr. v. Schooten in his commentary on Descartes's *Géométrie*, in the edition which appeared in Amsterdam in 1644.[9] This notation was used by Huygens and Mersenne in 1646 in their correspondence with each other,[10] and by Hudde in 1658.[11] Oughtred did

[1] C. I. Gerhardt, *Entdeckung der Differentialrechnung durch Leibniz*, Halle, 1848, pp. 53, 54.
[2] Cantor, *Geschichte der Mathematik*, Vol. II, 1892, p. 781.
[3] *Acta Eruditorum*, 1684 = Leibniz, *Werke*, Vol. V, 1858, pp. 220–226.
[4] Cantor, *op. cit.*, III, 1898, pp. 187, 188.
[5] Gerhardt, *op. cit.*, p. 35.
[6] *La géométrie de René Descartes*. Nouvelle Édition. Paris, 1886, p. 2.
[7] Chuquet, "Le triparty," *Bullettino Boncompagni*, Tomo XIII, Rome, 1880, p. 740.
[8] Harriot, *Artis analyticæ praxis*, London, 1631, p. 156.
[9] Matthiessen, *Grundzüge der Antiken u. Modernen Algebra*, Leipzig, 1878, p. 551.
[10] C. Huygens, *Œuvres*, T. I, La Haye, 1888, p. 24.
[11] Joh. Huddenii *Epist. I de reductione æquationum*, Amsterdam, 1658; Matthiessen, *op. cit.*, p. 374.

not use the modern exponents in any of his editions of his *Clavis mathematica* (London, 1631, 1648, 1652), but if Rigaud reproduced faithfully the notations in the original letters, it follows that Oughtred used positive integral exponents in his correspondence as early as 1642.[1] On Feb. 5, 1666–7, John Wallis wrote to John Collins, a proposed new edition of Oughtred's *Clavis* being under discussion: "It is true, that as in other things so in mathematics, fashions will daily alter, and that which Mr. Oughtred designed by great letters may now by others be designed by small; but a mathematician will, with the same ease and advantage, understand *Ac* and a^3 or *aaa*."[2] John Pell wrote r^2 and t^2 in a letter written in Amsterdam on Aug. 7, 1645.[3] Pascal made free use of positive integral exponents in several of his papers, particularly the *Potestatum numericarum summa*, 1654. G. Kinckhuysen used positive integral exponents in 1660, in his *Meet-Konst*, and in 1661 in his *Algebra*.[4]

[The February issue will conclude the discussion of the modern exponential notation and take up the unsuccessful attempts to create a theory of logarithms of negative numbers.]

ERRORS IN THE LITERATURE ON GROUPS OF FINITE ORDER.

By G. A. MILLER, University of Illinois.

Errors in General.

Although mathematics is an exact science, the mathematical literature is disfigured by numerous errors. Some of the most influential works fail along this line. In the preface to volume 1 of his *Zahlentheorie*, 1892, Paul Bachmann observes that Gauss' *Disquisitiones arithmeticæ*, 1801, was scarcely readable on account of the large number of annoying typographical errors.

Imperfect Proofs.—While typographical errors are annoying they are not the most serious errors. A much more annoying class of errors is due to imperfect proofs. The careless use of infinite series during the seventeenth and eighteenth centuries, without any inquiry as to convergence, is one of the most important instances of imperfect methods. In fact, in the early part of the nineteenth century there were only a few mathematicians who were sufficiently careful about this matter; so that Abel could properly write, in 1826, to his friend Holmboe, that most of the papers dealing with series were inexact.[5] Recently Study called atten-

[1] S. J. Rigaud, *Correspondence of Scientific Men of the Seventeenth Century*, Vol. I, Oxford, 1841, p. 63.

[2] Rigaud, *op. cit.*, Vol. I, p. 475.

[3] J. O. Halliwell, *Progress of Science in England*, London, 1841, p. 89.

[4] Kinckhuysen, *De Grondt der Meet-Konst*, De Haerlem, 1660; *Algebra ofte Stel-Konst*, De Haerlem, 1661.

[5] Niels Henrik Abel Memorial, Correspondance, 1902, p. 16.

tion to the great lack of precision in much of our recent geometrical production, referring to it as "indegesta moles geometrica."[1]

A third large class of errors is due to the oversight of an important step in an otherwise perfect proof. Many of the supposed proofs of the parallel axiom, starting with those of Ptolomy and Proclus, are of this nature. This is also true of many of the supposed proofs of the fundamental theorem of algebra, including those associated with such prominent names as d'Alembert, Euler, Lagrange, and Laplace.[2] Many recent examples of such errors are found in the literature relating to attempts to prove the great Fermat theorem.

Historical References.—Another large class of errors relates to historical notes. As a result of the recent very rapid progress in the development of our knowledge of the history of mathematics a very large number of the older mathematical references have been proved to be incorrect. Even some of the most recent and most reliable books disclose a lack of sufficient care in this respect. As instances of this kind, we may mention the erroneous statements regarding the early use of the term function, given both in the *Fundamental Concepts of Algebra and Geometry* by J. W. Young, 1911, page 194, and also in the *Monographs on Modern Mathematics* by J. W. A. Young, 1911, page 265.[3] It is, of course, generally unnecessary to give historical references in a mathematical work, but when they are given they should be correct.

Lists of Errors.—In 1904 Maillet proposed, as question number 2855 in *L'Intermédiaire des Mathématiciens*, the making of a list of errors committed by prominent deceased mathematicians. Maillet began this list by referring to errors committed by Ampère, Abel, Chasles, Legendre, Bertrand, Laplace, and others. Many others have contributed to this list but it is still very incomplete. It is, however, sufficiently extensive to exhibit many interesting questions with important caution signals, and it is to be hoped that this list will be gradually extended. A similar list relating to the errors committed by the great living mathematicians would probably be still more useful.

The question of errors in mathematical literature is especially important at the present time in view of their rapid increase on account of the numerous efforts to prove Fermat's theorem, and in view of the great encyclopedias of mathematics which are in course of publication. Both of these encyclopedias invite the mathematical world to coöperate in the effort to make advances and to banish falsehoods. For this purpose the German encyclopedia established a "Sprechsaal" in the *Archiv der Mathematik und Physik* which was transferred in 1907 to the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, while the French edition publishes a "Tribune publique" in connection with the regular numbers of the encyclopedia. It is expected to embody all the valuable features of the "Tribune publique" in the "Additions et Modifications" which are to appear near the end of each volume. In view of the extensive and permanent circulation

[1] Study, *Archiv der Mathematik und Physik*, volume 18 (1911), p. 175.

[2] Cf. *Encyclopédie des Sciences Mathématiques*, tome 1, vol. 2, p. 193.

[3] This error seems to have been started by d'Alembert. It has been corrected in the *Encyclopédie des Sciences Mathématiques*, tome II, vol. 1, p. 3.

of these encyclopedias it is extremely important that they should be as reliable as possible, and it is to be hoped that their appeal for public coöperation will lead to more and more valuable responses. It is almost as important that matters of no value be excluded from such standard works as it is that all matters which exhibit real progress should be included.

First Aid to Truth.—For the first correction of historical errors the *Bibliotheca Mathematica* affords an excellent medium. This journal has already rendered most valuable services by directing attention to the large number of errors in some of the most popular works on the history of mathemaics, and by exposing these errors in a fearless manner. It should be remembered that errors are weeds and that they tend to impede the growth of the truth. It is one of the main duties of the true scholar to assist in their destruction, since this is essential for a perfect harvest. The man who fearlessly and unselfishly exposes errors, especially when they appear in the best books, is thereby rendering a great service, which is frequently not sufficiently appreciated.

Reviews constitute the most important single instrument to combat errors. Unfortunately reviewers do not always use this instrument wisely. In many cases this may be due to a lack of self-confidence on the part of the reviewer. In other cases it is doubtless due to the inclination to write a review which may please the author of the work under review. A conscientious reviewer cannot escape the sense of duty to the public and to truth. If we could have more reviews like those which Study, of Bonn, writes, we should doubtless have a more enlightened public and a larger number of careful authors.

Students of mathematics should read more semi-mathematical literature, such as these reviews, and the historical or philosophical mathematical literature. Many new viewpoints come to us in this way. Errors due to eminent mathematicians furnish an especially valuable topic of a semi-mathematical nature, and even if a student reads reviews of books and of articles mainly from a sense of curiosity as to how the work of those whom he knows is regarded by competent critics, it is very much better than if he reads nothing outside of those things which bear directly on his regular work. It should be remembered, however, that it is very easy to discover little errors in a great man's work, just as it is easy to make slight extensions to such work. The correction of really big and important errors is the work of a big man. A man discloses his caliber by the way he corrects errors just as much as by his other publications.

Errors in the Early Literature of Group Theory.

Galois was the first to enter deeply into the theory of groups. In his efforts to develop a general theory of equations he directed attention to the important concept of simple groups and proved that the icosahedral group is the smallest simple group of composite order. He also studied a very important infinite system of simple groups, known as the general modular groups, of order $\frac{1}{2}p(p^2 - 1)$ and of degree $p + 1$, $p > 3$. In one of his earliest papers, published in the

Bulletin des Sciences Mathématiques de M. Férussac, 1830, page 271, he asserts that this modular group can be represented as a substitution group of degree p only when $p = 5$. Later he corrected his mistake by observing that this representation is also possible when $p = 7$ or 11, but for no larger value of p. A recent proof of this important theorem is due to Frobenius, who also observes that the modular group is the only simple group of order $\frac{1}{2}p(p^2 - 1)$,[1] $p > 3$.

Errors of Cauchy.—Cauchy was the first who wrote extensively on the theory of substitution groups, and he is commonly called the founder of this theory in view of his many valuable contributions along this line. Unfortunately, the number of errors in his writings is very large, and most of them are repeated in the published parts of his complete works. As these works are brought out under the auspices of the Paris Academy of Sciences, one might have expected footnotes calling attention to the errors which are so distracting and which impair so seriously the usefulness of the work; especially since the publication is proceeding very slowly, having commenced about thirty years ago. Not only do these writings involve many erroneous statements but they also contain obscurities which later developments have completely cleared away. Those who edit collected works do not always take sufficient care to avoid perpetuating the errors and obscurities of a writer.

Cauchy began the construction of fairly long lists of substitution groups which are supposed to contain all the possible substitution groups on six or a smaller number of letters. Such lists were based by Cauchy upon an insufficient knowledge of the theory of substitutions and hence they were incomplete. Very many later writers had a similar experience and the efforts to enumerate all the possible substitutions groups of certain degrees have led to a very large number of errors in the literature of group theory, especially on the part of Kirkman, Askwith, and Cayley. One of the most important instances of false theorems announced by Cauchy is the one which states that every simply transitive group of degree $p + 1$, p being a prime number, is imprimitive.[2]

Errors of Cayley.—Cayley's writings on group theory are scarcely more accurate than those of Cauchy. In volume 1 of the *American Journal of Mathematics*, page 51, Cayley states that there are three groups of order 6, giving as examples, the non-cyclic group and the cyclic group in two different forms. It seems very strange that Cayley should have made a mistake regarding such a simple matter more than twenty years after he began to study group theory. About thirteen years later Cayley attempted to enumerate the substitution groups[3] whose degrees do not exceed 8 and in this work the number of errors is very large, as has been observed by various writers. These errors have been repeated in the *Collected Mathematical Papers*, volume XIII, 1897, and the very evident error in reference to the groups of order 6 is also repeated in volume X, 1896.

Errors of Jordan.—Jordan did very much pioneer work in group theory and he attacked some of the most difficult problems. While the number of errors in

[1] Frobenius, *Berliner Sitzungsberichte*, 1902, page 351.
[2] Cauchy, Paris, *Comptes Rendus*, volume 21 (1845), page 1199.
[3] These enumerations appeared in *Quarterly Journal of Mathematics*, volume 25, 1891.

his works is considerable it is perhaps not larger than one could reasonably expect from the difficulty and newness of the problems. In particular, as early as 1878 he made the first attempt to determine all the finite collineation groups in three variables, but he failed to find the two important groups of orders 168 and 360, which were discovered later by Klein and Valentiner respectively. He also attempted to determine all the primitive substitution groups which can be represented on no more than 17 letters, but in this enumeration he also omitted a number of important groups. In particular he found only 14 out of 22 primitive groups of degree 16.

Errors in Definitions.—The fact that early writers generally did not use a precise definition of group is a source of considerable confusion.[1] Even in such a good work as Klein's *Ikosaeder*, page 5, we read that a set of operations constitute a group when the composition of two of them gives one of the set. It is, of course, true that such a general definition may be used, and one finds authority for it in the French encyclopedia, tome 1, volume 2, page 243. In fact, the term group is defined in two very distinct ways in this standard work of reference. A commonly accepted definition is found on page 576 of the first volume. It seems to the writer unfortunate that two distinct definitions of this widely used term should appear in a work of reference, since this may tend to perpetuate its use in a vague manner by careless writers.

A Few Recent Errors in Group Theory.

Errors in the Encyclopædias.—A queer historical error occurs under the word "group" in the latest edition of the *Encyclopædia Britannica*. It is stated here that the mathematical technical sense of the word group is not older than 1870. If we recall that Cayley used the term group with its technical meaning in the heading of a number of his articles, beginning as early as 1854; and that Kirkman and Sylvester also used it about 1860, it appears very strange that an Englishman should have made the above statement. Numerous other errors relating to the history of mathematics occur in the latest edition of this popular work of reference but none of them appears less excusable than the one which has just been noted.

That the great French mathematical encyclopedia is not free from errors in its article on group theory may readily be seen from the many corrections which have appeared in the "Tribune publique." The German mathematical encyclopedia devoted such a small amount of space to the subject of finite groups that one could scarcely expect many errors. There are, however, also several somewhat serious errors in this work. Perhaps the most important one occurs on page 221, where it is stated, in substance, that the holomorph of a group is its group of isomorphisms, although the term holomorph is not used.

A very important recent general article on groups of finite order appeared in the latest edition of *Pascal's Repertorium*. This article is also unusually

[1] The present abstract definition is due to Germans; especially to Kronecker (1870), Weber (1882), and Frobenius (1887).

free from errors, notwithstanding its broad scope. One of the most important errors in it occurs on page 229, where the fundamental theorem, that with each regular group there may be associated a similar regular group in the same letters such that each of these groups is composed of all the substitutions in these letters which are commutative with every substitution of the other, is credited to Cayley (1878). As a matter of fact Jordan proved this theorem in his thesis published in 1860 and republished in *L'École Polytechnique* during the following year.

Error of Weber.—A very interesting error occurs in volume 2, page 54, of the first edition of Weber's *Lehrbuch der Algebra*. It is stated here that the most important example of a commutative group is furnished by the natural numbers when they are combined by ordinary multiplication. As a matter of fact these numbers do not form a group at all, as Weber himself observes in the second edition of his classic algebra. This most important example of a commutative group is therefore no group. In this connection we may refer to a historic misstatement made by Weber several years ago and repeated by many others. In his biography of Kronecker, which appeared at the beginning of volume 43 of the *Mathematische Annalen*, Weber states that Kronecker mentions Galois for the first time in a letter to Dirichlet in 1856. On the other hand, Kronecker had referred several times to Galois' theory in an article which was read several years earlier and appeared in the *Berichte* of the Berlin Academy for 1853.

Inconsistent Definitions.—An instance where the technical and the non-technical meaning of the word group are confused may deserve notice here, since it occurs in a very useful reference work; namely, in volume I of the *Subject Index* of the Royal Society of London Catalogue of Scientific Papers, which was published in 1908. On page 66 of this index we find a reference to Cockle's papers on "Vanishing Groups." This reference appears under the technical subject groups, but the papers by Cockle refer entirely to groups in the non-technical sense and have therefore nothing to do with groups as a technical concept.

The fact that the same name is assigned to two distinct groups in two well-known works of reference is a source of some confusion. In volume I, page 238, of the latest edition (1910) of *Pascal's Repertorium*, the symmetric group of order 6 is called the *anharmonic group*, while Capelli uses the same name for an entirely different group in the most recent edition of his algebra, entitled, *Istituzioni di analisi algebrica*, 1909, page 111. It may also be observed that some writers use the term degree for what is commonly known as the order of a group, and that the term metacyclic is used with different meanings.

Lack of References.—The literature of group theory has been made unnecessarily complicated by a lack of a proper appreciation of the importance of due references. In fact, some writers have not only failed to give proper references but have not even taken the trouble to find out whether their papers contained any new results before offering them for publication. In many cases such errors are not noted even in the reviews which appear in such good works as the *Jahrbuch ueber die Fortschritte der Mathematik*.

As an instance of this kind we may cite an article by Netto which appeared in *Crelle's Journal*, volume 128 (1905), page 243. We find in this article a development of the tetrahedral group according to an abstract definition which has been well known for more than half a century, and yet no references are given by the author of the article in question or by the reviewer in the *Fortschritte*. The article contains a number of other things which had been published several years before this article appeared. An instance of an article which contained nothing new and yet appeared in the excellent *Mathematische Annalen*, without proper reference, may be found by consulting this journal, volume 60 (1905), page 319.

Errors of Judgment.—There is a class of indirect errors to which it may be desirable to refer here; namely, those which arise from undue emphasis on some particular phase of the subject or on the work of some particular man. According to the writer's opinion Burnside's *Theory of Groups* errs along this line by quoting Hölder's work relatively too frequently. In fact, these references are not always correct. Even in the second edition, page 39, we are told that the symbol for quotient group was introduced by Hölder, although this symbol had been employed by Jordan at a much earlier date. As other instances of undue emphasis in important publications with respect to the work of one man, we may cite the references to Frobenius in the excellent article on groups contained in *Pascal's Repertorium* and the references to the present writer in the French encyclopedia article on this subject.

Conclusion.—It may be added that the above are only a few typical examples of the many errors which have been committed by some of those who have contributed much to the development of group theory. A complete collection of such errors, with proper explanations, would make a volume. Some of these errors have not had an unmitigatedly baneful effect, as they inspired young investigators with confidence, when these found that they could correct mistakes in the works of eminent mathematicians. The contributions to mathematics, due to the inspiration received from the discovery of errors in the works of eminent men, have been of great value to our science; but the beauty and insight furnished by correct and fundamental results are still more inspiring and are much more worthy objects of our endeavors.

ON THE REMAINDER TERM IN A CERTAIN DEVELOPMENT OF $f(a + x)$.*

By R. D. CARMICHAEL, Indiana University.

In the *Annals of Mathematics*, Vol. 5, No. 4, July, 1904, Mr. S. A. Corey gave an interesting and important development of $f(a + x)$, and showed that his formula was admirably adapted to the numerical computation of complicated

* Read before the American Mathematical Society, February 29, 1908.

definite integrals. The object of this paper is to find the "remainder term" in Corey's series* and incidentally to verify the series.

Putting the expansion in a form somewhat different from that of Corey and writing after it the remainder term R_{2n+2}, we shall verify the series and find the value† of R_{2n+2}:

$$
\begin{aligned}
f(a+x) = f(a) + \frac{x}{m}\Big[f'\Big(a+\frac{x}{m}\Big) + f'\Big(a+\frac{2x}{m}\Big) + \cdots + f'\Big(a+\frac{m-1}{m}x\Big) \\
+ f'(a+x)\Big] - \frac{x}{m\cdot 2}[f'(a+x) - f'(a)] - \frac{B_1 x^2}{m^2\cdot 2!}[f''(a+x) - f''(a)] \\
+ \frac{B_2 x^4}{m^4\cdot 4!}[f^{(\mathrm{iv})}(a+x) - f^{(\mathrm{iv})}(a)] - \cdots \\
+ (-1)^n \frac{B_n x^{2n}}{m^{2n}\cdot (2n)!}[f^{(2n)}(a+x) - f^{(2n)}(a)] + R_{2n+2}.
\end{aligned} \tag{1}
$$

Here m is a positive integer and $B_1, B_2, \cdots$ are the well-known Bernoulli numbers:

$$
B_1 = \tfrac{1}{6}, \quad B_2 = \tfrac{1}{30}, \quad B_3 = \tfrac{1}{42}, \quad B_4 = \tfrac{1}{30}, \quad B_5 = \tfrac{5}{66}, \quad \cdots.
$$

Throughout the discussion it is assumed that $f(x)$ and its derivatives employed exist and are continuous over the range a to $a+x$.

When $r \leqq m$ one sees from Taylor's development with remainder term that

$$
\begin{aligned}
f'\Big(a+\frac{rx}{m}\Big) = f'(a) + \frac{rx}{m}f''(a) + \frac{r^2x^2}{2!\cdot m^2}f'''(a) + \cdots \\
+ \frac{r^{2n}x^{2n}}{m^{2n}\cdot (2n)!}f^{(2n+1)}(a) + \frac{r^{2n+1}x^{2n+1}}{m^{2n+1}\cdot (2n+1)!}f^{(2n+2)}(a+\varphi_r x),
\end{aligned} \tag{2}
$$

where $0 < \varphi_r < 1$. If in this equation r is put successively equal to $1, 2, 3, \cdots, m$, and the resulting equation in each case is multiplied by x/m, we have by addition:

$$
\begin{aligned}
\frac{x}{m}\Big[f'\Big(a+\frac{x}{m}\Big) + f'\Big(a+\frac{2x}{m}\Big) + \cdots + f'\Big(a+\frac{m-1}{m}x\Big) + f'(a+x)\Big] \\
= xf'(a) + x^2 f''(a)\Big[\frac{1+2+3+\cdots+m}{m^2}\Big] + \frac{x^3 f'''(a)}{2!}\Big[\frac{1^2+2^2+3^2+\cdots+m^2}{m^3}\Big] \\
+ \cdots + \frac{x^{2n+1}f^{(2n+1)}(a)}{(2n)!}\Big[\frac{1^{2n}+2^{2n}+3^{2n}+\cdots+m^{2n}}{m^{2n+1}}\Big] \\
+ \frac{x^{2n+2}}{m^{2n+2}\cdot (2n+1)!}[f^{(2n+2)}(a+\varphi_1 x) + 2^{2n+1}f(a+\varphi_2 x) + \cdots \\
+ m^{2n+1}f^{(2n+2)}(a+\varphi_m x)].
\end{aligned} \tag{3}
$$

* Mr. Corey gave a partial solution of this problem in the AMERICAN MATHEMATICAL MONTHLY, Vol. 14, Nos. 6–7, pp. 131–135; June–July, 1907.

† The value in final form is given in equation (13) below.

Again from Taylor's development it follows that

$$f^{(s)}(a+x)-f^{(s)}(a)=xf^{(s+1)}(a)+\frac{x^2}{2!}f^{(s+2)}(a)+\cdots + \frac{x^{2n-s+1}}{(2n-s+1)!}f^{(2n+1)}(a)+\frac{x^{2n-s+2}}{(2n-s+2)!}f^{(2n+2)}(a+\theta_s x), \tag{4}$$

where $0<\theta_s<1$. We shall again form a series of equations and take their sum. Let $s=1$ and multiply the resulting equation by $-x/m\cdot 2$. Then take $s=2$ and multiply the resulting equation by $-B_1x^2/m^2\cdot 2!$. Then take $s=4$ and multiply by $B_2x^4/m^4\cdot 4!$, and so on. By addition we should finally have the following:

$$\begin{aligned}
&-\frac{x}{m\cdot 2}[f'(a+x)-f'(a)]-\frac{B_1x^2}{m^2\cdot 2!}[f''(a+x)-f''(a)]+\frac{B_2x^4}{m^4\cdot 4!}\\
&\times[f^{(\mathrm{iv})}(a+x)-f^{(\mathrm{iv})}(a)]-\cdots+(-1)^n\frac{B_nx^{2n}}{m^{2n}\cdot(2n)!}[f^{(2n)}(a+x)-f^{(2n)}(a)]\\
&=x^2f''(a)\left[-\frac{1}{m\cdot 2}\right]+x^3f'''(a)\left[-\frac{1}{2m\cdot 2!}-\frac{B_1}{m^2\cdot 2!}\right]+x^4f^{(\mathrm{iv})}(a)\\
&\times\left[-\frac{1}{2m\cdot 3!}-\frac{B_1}{m^2\cdot 2!\cdot 2!}\right]+x^5f^{(\mathrm{v})}(a)\left[-\frac{1}{2m\cdot 4!}-\frac{B_1}{m^2\cdot 2!\cdot 3!}+\frac{B_2}{m^4\cdot 4!}\right]\\
&+x^{2n+2}\left[-\frac{1}{2m\cdot(2n+1)!}f^{(2n+2)}(a+\theta_1x)-\frac{B_1}{m^2\cdot 2!(2n)!}f^{(2n+2)}(a+\theta_2x)\right.\\
&\left.+\frac{B_2}{m^4\cdot 4!(2n-2)!}f^{(2n+2)}(a+\theta_4x)-\cdots+(-1)^n\frac{B_n}{m^{2n}\cdot(2n)!\cdot 2!}f^{(2n+2)}(a+\theta_{2n}x)\right].
\end{aligned} \tag{5}$$

If now we add to the sum of the first members of (3) and (5) the terms $f(a)$ and R_{2n+2} we reproduce the second member of (1). Hence by (1),

$$f(a+x)=f(a)+\text{second member of (3)}+\text{second member of (5)}+R_{2n+2}.$$

In order to simplify the sum of these second members, let us consider the coefficients of the two expressions $x^{2r}f^{(2r)}(a)$ and $x^{2r+1}f^{(2r+1)}(a)$. In the sum of the second members of (3) and (5) the coefficient of $x^{2r}f^{(2r)}(a)$ is

$$\frac{1^{2r-1}+2^{2r-1}+\cdots+m^{2r-1}}{m^{2r}\cdot(2r-1)!}-\frac{1}{2m\cdot(2r-1)!}-\frac{B_1}{m^2\cdot 2!(2r-2)!}+\frac{B_2}{m^4\cdot 4!(2r-4)!}-\cdots+(-1)^{r-1}\frac{B_{r-1}}{m^{2r-2}\cdot 2!(2r-2)!}. \tag{6}$$

The coefficient of $x^{2r+1}f^{(2r+1)}(a)$ is

$$\frac{1^{2r}+2^{2r}+\cdots+m^{2r}}{m^{2r+1}\cdot(2r)!}-\frac{1}{2m\cdot(2r)!}-\frac{B_1}{m^2\cdot 2!(2r-1)!}+\frac{B_2}{m^4\cdot 4!(2r-3)!}$$
$$-\cdots+(-1)^{r-1}\frac{B_{r-1}}{m^{2r-2}\cdot(2r-2)!2!}+(-1)^r\frac{B_r}{m^{2r}\cdot(2r)!}. \tag{7}$$

But by a theorem due to Bernouilli we have

$$1^r+2^r+3^r+\cdots+m^r=\frac{m^{r+1}}{r+1}+\frac{1}{2}m^r+\frac{r!}{(r-1)!2!}B_1m^{r-1}$$
$$-\frac{r!}{(r-3)!4!}B_2m^{r-3}+\frac{r!}{(r-5)!6!}B_3m^{r-5}\cdots, \tag{8}$$

the second member ending with

$$(-1)^{\frac{r-2}{2}}\frac{r!}{1!\,r!}B_{\frac{r}{2}}m,$$

if r is even and with

$$(-1)^{\frac{r-3}{2}}\frac{r!}{2!\,(r-1)!}B_{\frac{r-1}{2}}m^2,$$

if r is odd. (See Chrystal's Algebra, vol. II, page 209 (2).)

Replacing r by $2r-1$ and substituting in (6) we have for the coefficient of $x^{2r}f^{(2r)}(a)$ an expression which readily reduces to $1/(2r)!$. If we replace r by $2r$ and substitute in (7) we find that the coefficient of $x^{2r+1}f^{(2r+1)}(a)$ reduces to $1/(2r+1)!$. Hence the relation $f(a+x)=f(a)+$ second member of (3) $+$ second member of (5) $+R_{2n+2}$ will reduce to

$$f(a+x)=f(a)+xf'(a)+\frac{x^2}{2!}f''(a)+\cdots+\frac{x^{2n+1}}{(2n+1)!}f^{(2n+1)}(a)$$
$$+\frac{x^{2n+2}}{m^{2n+2}\cdot(2n+2)!}[f^{(2n+2)}(a+\varphi_1x)+2^{2n+1}f^{(2n+2)}(a+\varphi_2x)+\cdots$$
$$+m^{2n+1}f^{(2n+2)}(a+\varphi_mx)]+x^{2n+2}\Big[-\frac{1}{2^m\cdot(2n+1)!}f^{(2n+2)}(a+\theta_1x)$$
$$-\frac{B_1}{m^2\cdot 2!\,(2n)!}f^{(2n+2)}(a+\theta_2x)+\frac{B_2}{m^4\cdot 4!\,(2n-2)!}f^{(2n+2)}(a+\theta_4x)$$
$$-\cdots+(-1)^n\frac{B_n}{m^{2n}\cdot(2n)!\,2!}f^{(2n+2)}(a+\theta_{2n}x)\Big]+R_{2n+2}. \tag{9}$$

Equation (9) is identical with Taylor's development up to the term containing x^{2n+1}; therefore the remainder after this term in (9) may be equated to the well-known remainder

$$\frac{x^{2n+2}}{(2n+2)!}f^{(2n+2)}(a+\theta x),\qquad 0<\theta<1,$$

after the corresponding term in Taylor's series. This readily yields the value of R_{2n+2} which is sought:

$$
\begin{aligned}
R_{2n+2} = {} & \frac{x^{2n+2}}{(2n+2)!} f^{(2n+2)}(a+\theta x) - \frac{x^{2n+2}}{m^{2n+2}\cdot(2n+2)!}[f^{(2n+2)}(a+\varphi_1 x) \\
& + 2^{2n+1} f^{(2n+2)}(a+\varphi_2 x) + \cdots + m^{2n+1} f^{(2n+2)}(a+\varphi_m x)] \\
& + x^{2n+2}\left[\frac{1}{2m\cdot(2n+1)!} f^{(2n+2)}(a+\theta_1 x) + \frac{B_1}{m^2\cdot 2!\,(2n)!} f^{(2n+2)}(a+\theta_2 x)\right. \\
& - \frac{B_2}{m^4\cdot 4!\,(2n-2)!} f^{(2n+2)}(a+\theta_4 x) + \cdots \\
& \left. - (-1)^n \frac{B_n}{m^{2n}\cdot(2n)!\,2!} f^{(2n+2)}(a+\theta_{2n} x)\right].
\end{aligned} \tag{10}
$$

In order to simplify this result we shall make use of the following lemma:

If $a_1, a_2, \cdots, a_i$ are all positive and $\varphi_1, \varphi_2, \cdots, \varphi_i$ are all positive and less than unity, then there exists a positive φ, also less than unity, such that

$$
\begin{aligned}
& a_1 F(a+\varphi_1 x) + a_2 F(a+\varphi_2 x) + \cdots + a_i F(a+\varphi_i x) \\
& \qquad = (a_1 + a_2 + \cdots + a_i) F(a+\varphi x),
\end{aligned} \tag{11}
$$

provided that the function F is continuous.

This may be proved thus: If A and B are the greatest and least values of F in the first member of (11), then is this member increased by replacing F by A and decreased by replacing F by B. Hence, some value, say $F(a+\varphi x)$, between these limits may replace each F in (11) without altering the value of the expression. Hence the theorem. Evidently the proposition also holds if all the a's are negative.

Applying this lemma to equation (10), we may replace each $f^{(2n+2)}$ in the first brackets by $f^{(2n+2)}(a+\varphi x)$. The coefficient of $x^{2n+2} f^{(2n+2)}(a+\varphi x)$ then is

$$
-\frac{1}{m^{2n+2}\cdot(2n+2)!}(1 + 2^{2n+1} + 3^{2n+1} + \cdots + m^{2n+1}).
$$

Changing the form of this coefficient by (8) and substituting in (10) we have

$$
\begin{aligned}
R_{2n+2} = {} & x^{2n+2}\left[\frac{1}{(2n+2)!} f^{(2n+2)}(a+\theta x) - \frac{1}{(2n+2)!} f^{(2n+2)}(a+\varphi x)\right. \\
& - \frac{1}{2m\cdot(2n+1)!} f^{(2n+2)}(a+\varphi x) - \frac{B_1}{m^2\cdot 2!(2n)!} f^{(2n+2)}(a+\varphi x) + \cdots \\
& \left. + (-1)^n \frac{B_n}{m^{2n}\cdot(2n)!\,2!} f^{(2n+2)}(a+\varphi x) + \frac{1}{2m\cdot(2n+1)!} f^{(2n+2)}(a+\theta_1 x)\right.
\end{aligned} \tag{12}
$$

$$+\frac{B_1}{m^2\cdot 2!(2n)!}f^{(2n+2)}(a+\theta_2 x)-\frac{B_2}{m^4\cdot 4!(2n-2)!}f^{(2n+2)}(a+\theta_4 x)+\cdots$$

$$-(-1)^n\frac{B_n}{m^{2n}\cdot(2n)!\,2!}f^{(2n+2)}(a+\theta_{2n}x)\Big].$$

It will be noticed that this remainder may have its terms grouped into two sets whose terms correspond, term of one to term of the other, and such that all the terms of one set have positive, and all those of the other have negative, coefficients. Then by the lemma already proved, the f's in the two sets may be replaced respectively by $f^{(2n+2)}(a+\varphi_1 x)$ and $f^{(2n+2)}(a+\varphi_2 x)$ where φ_1 and φ_2 are positive and less than unity. Making these changes in (12) we have the following final value for the remainder term:

$$R_{2n+2}=x^{2n+2}\Big[\frac{1}{(2n+2)!}+\frac{1}{2m\cdot(2n+1)!}+\frac{B_1}{m^2\cdot 2!(2n)!}+\frac{B_2}{m^4\cdot 4!(2n-2)!}$$

$$+\cdots+\frac{B_{n-1}}{m^{2n-2}\cdot(2n-2)!\,4!}+\frac{B_n}{m^{2n}\cdot(2n)!\,2!}\Big] \tag{13}$$

$$\times[f^{(2n+2)}(a+\varphi_1 x)-f^{(2n+2)}(a+\varphi_2 x)].$$

NEW BOOKS.

Under this heading will be listed all new books received, which pertain to the general subject of mathematics as related to the collegiate and advanced secondary fields. So far as possible either short descriptive notices or more extended reviews will be given, according to the judgment of the committee having this department in charge. The chairman of this committee is Professor W. H. Bussey, of the University of Minnesota, Minneapolis, Minn., and the other members are Professor C. H. Ashton, University of Kansas, Professor W. C. Brenke, University of Nebraska, and Professor L. C. Karpinski, University of Michigan.

Publishers desiring to use this means of conveying to teachers of mathematics information concerning new books will please forward such books to the chairman of the committee.

TWO NEW BOOKS ON THE CALCULUS.

By W. H. BUSSEY, University of Minnesota.

Differential and Integral Calculus. An introductory course for colleges and engineering schools. By LORRAIN S. HULBURT. Longmans, Green, and Co., New York, 1912. xviii+481 pages.

The Calculus. By ELLERY WILLIAMS DAVIS, assisted by WILLIAM CHARLES BRENKE. Edited by EARLE RAYMOND HEDRICK. The Macmillan Company, New York, 1912. xx+384+63 pages.

In many American colleges and universities, differential and integral calculus are still taught as separate subjects. It is the traditional method. The more modern tendency is to teach a course in "The Calculus," the two branches of the subject being more or less closely correlated according to the text book used. Whether a teacher gives a course of the one kind or the other depends sometimes upon his individual preference, and sometimes upon peculiar conditions in the college or in the department of mathematics. As a usual thing, the same book is not well adapted to the two kinds of courses. The two books which are the subject of this review represent the two methods of teaching the subject.

Hulburt's Calculus keeps differential and integral calculus quite separate. The Davis Calculus correlates the two branches of the subject. It is the opinion of the reviewer that each is an excellent book of its kind. The teacher who is looking for a new text book has a choice between two widely different types. They differ in point of view as well as in method. The fundamental idea of the Davis Calculus is that the derivative is the rate of change of one quantity with respect to another. The whole subject is evolved from the idea of a rate as a limit. The very first paragraph of the book defines the calculus as the study of the relations between interdependent quantities with special reference to their rates of change. The derivative as the slope of a curve and the derivative as speed are regarded as special cases of the derivative as rate of change. In the Hulburt Calculus, on the other hand, the interpretation of the derivative as a rate appears for the first time on page 103, in the chapter on "The derivative as velocity." For the first eight chapters, the derivative is interpreted as the slope of the curve represented by the function. The tendency of the Davis Calculus is to correlate the various parts of the subject, while that of Hulburt's Calculus is to keep them separate.

Another fundamental difference between the books is illustrated by the two ways in which the authors introduce the indefinite integral. In the Hulburt book, the indefinite integral of a function is an abstract thing. The problem is that of finding a function whose derivative is given. The student learns the technique of abstract integration, and then he studies its applications to geometry and kinematics. The advantage of this method is that difficulties of different kinds are met and overcome separately. In the Davis book, on the other hand, the indefinite integral is given a concrete interpretation when it is introduced. The problem is a real problem, namely that of finding the amount of a variable

quantity whose rate of change is known. The work is vitalized by having a meaning attached to it from the beginning. That is an advantage. The student sees what it is for. The disadvantage is that he has the difficulties of interpretation and the difficulties of technique to meet at the same time. This is a typical example of an important difference between the two books. Each type of book has its own peculiar advantages and disadvantages.

Hulburt's Calculus.

Hulburt's Calculus may be thought of as divided into four parts. Book I (174 pages) deals entirely with differential calculus, Books II and III (118 pages) deal entirely with integral calculus. Books IV and V (127 pages) discuss the following topics in both differential and integral calculus, which were omitted in the earlier part of the book: partial and total derivatives; multiple integrals; centers of mass; the theorems of Taylor and Maclaurin; De Moivre's theorem and hyperbolic functions; integration of rational fractions; and envelopes. There is some correlation of the two branches of the calculus in the working out of these topics, but not much. Integration is used in the study of the hyperbolic functions, though it is needed for some of the examples given in connection with envelopes. Book VI (31 pages) is an introduction to ordinary differential equations.

Book I has an introduction dealing with functional symbols, kinds of functions, and graphs of functions with special reference to their discontinuities. Chapter I deals more extensively than is usual with limits and elementary means of finding limits. A strong feature of the book is the way in which it connects with analytic geometry. The derivative is defined in Chapter III. It is immediately interpreted as the slope of the curve represented by the function, and, except for the practical problems in maxima and minima, the applications given in the first eight chapters are all to analytic geometry, especially to curve tracing. Especially worthy of note is the paragraph in Chapter I which explains the distinction between the limit of $f(x)$ when $x \doteq a$ and the value of $f(x)$ when $x = a$. In Chapter VI, the author devotes four pages to the graphical method of finding approximately the roots of algebraic and transcendental equations.

Logarithmic and exponential functions are differentiated for the first time in Chapter VIII. A few pages are given to the interesting compound interest law. Differentials are introduced in a satisfactory way in this chapter. Up to this point, the author has used the $D_x y$ notation for derivative. His remarks in introducing the new notation are interesting: "It is evident that differentials might have been introduced much earlier in the course—in fact, at any stage after the definition of the derivative. The reader must not infer that they are introduced here because any peculiar advantage is to accrue from their use in the following pages. The differential notation possesses few advantages over the notation that we have employed up to this point, and throws no new light upon any of the problems of mathematics. Indeed it has difficulties of its own and can produce confusion of mind in the beginner. But it is a part of our mathe-

matical inheritance from the early days of the calculus, and is still widely used, especially by writers on applied mathematics. Because, then, of their use in modern text books, and for this reason only, the student must become familiar with differentials. In the following pages, therefore, we shall frequently give the differential formulæ along with the derivative formulæ."

Up to this point, the derivative has been interpreted only as the slope of a curve. Chapter IX is entitled "The derivative as velocity." Applications to simple harmonic motion are given.

These first nine chapters constitute a good short course (114 pages). The student will know what differential calculus is good for, even if he never reads farther. The next seven chapters are short, They deal with applications to geometry and kinematics. The last chapter in Book I takes up the law of the mean and indeterminate forms. Partial and total derivatives occur in Book IV. They can be taken up as soon as Book I has been finished, if it is so desired. They are placed in Book IV because an introduction to solid analytic geometry is given in Book III. The theorems of Taylor and Maclaurin and their applications are also in Book IV. They can be taken up before integral calculus is begun if it is desired.

Integral calculus begins in Book II. The first principles of integration, including the fundamental formulæ and the simple devices for bringing the integrand into a form to which one of them will apply, are given in 15 pages. The author then explains how a table of integrals is used, and he suggests that the student buy one. There is no such table in the book. The table constantly referred to in the text is that of B. O. Peirce. The author believes that the student should use such a table freely, so as to have more time to spend on the applications of integral calculus. Nevertheless, he realizes that the student must know some of the devices for finding integrals if he is to be able to use the table intelligently. So he puts in eight pages on special methods of integration, including integration by parts and by algebraic and trigonometric substitution.

All this work in integral calculus takes just 26 pages. The next two chapters (23 pages) are devoted to applications of indefinite integrals to plane geometry and kinematics. Then comes Chapter XXII, dealing with definite integrals and their use in finding lengths of curves and areas under curves. It is in Chapter XXIII that the definite integral is thought of primarily as the limit of a sum. The author gives a geometric proof of the fundamental theorem that the sum $\sum_a^b f(x)\Delta x$ has a definite limit when $f(x)$ is real, single valued, and continuous throughout the interval $a \leqq x \leqq b$.

Book II closes with applications of definite integrals to solids and surfaces of revolution. Book III contains an introduction to solid analytic geometry, and applications of definite integrals to volumes and areas. Only single integration is used. Multiple integrals and their applications are found in Book IV, after partial derivatives. The miscellaneous topics in Book V have already been mentioned. Book VI is an introduction to ordinary differential equations.

Answers to all problems are given at the end of the book.

The Davis Calculus.

The Davis Calculus is the first of a new series of mathematical books to be published by The Macmillan Company under the general editorship of Professor E. R. Hedrick, of the University of Missouri.

The first paragraph of the preface sounds the key-note of the book. "The significance of the calculus, the possibility of applying it in other fields, its usefulness, ought to be kept constantly and vividly before the student during his study of the subject, rather than to be deferred to an uncertain future." The subject is made concrete from the beginning. The derivative is defined as the limit of the quotient of two infinitesimal differences Δy and Δx, but it always means the instantaneous rate of change of the variable y with respect to x. The meaning is made clear before the formal definition is given. Applications to geometry, kinematics, maxima and minima, etc., take up most of the first 90 pages. Not many of these pages are given to the technique of differentiation, because only algebraic functions are used. The derivatives of logarithmic, exponential, and trigonometric functions are taken up for the first time on page 130.

The differential notation is used from the beginning of the book. Differentials are defined in a satisfactory way in the third chapter. In leading up to this subject, the author remarks that, although the derivative is the limit of a fraction rather than a fraction, it may be thought of as a fraction by providing it with a convenient denominator, just as any other quantity may be thought of as a fraction if provided with a convenient denominator. The way in which the author makes this clear to the student is very ingenious. It certainly makes differentials concrete things.

In Chapter V, the indefinite integral is introduced in connection with the problem of the reversal of rates. The idea of rate is very much in evidence in the first applications of integrals to the finding of lengths of curves and areas under curves. In fact the fundamental summation formula which says that a definite integral is the limit of a sum is obtained from the interpretation of an indefinite integral as a reversed rate. A little later, the author remarks that "the greater number of integrations appear more naturally as limits of sums than as reversed rates," and he adds this foot note: "It is really a waste of time to discuss at great length which fact about integrals is used as a definition, and which one is proved; to satisfy the demand for formal definition, the integral may be defined in either way,—as a limit of a sum, or as a reversed differentiation. The important fact is that the two ideas coincide, which is the fact stated in the summation formula." The work on the definite integral as the limit of a sum is preceded by a few pages devoted to elementary methods of approximate summation as distinguished from the exact summation given by definite integrals. Applications of Cavalieri's theorem and the prismoid formula are included in this chapter.

One of the good things about this book is that the first five chapters (130

pages) contain so many useful and interesting applications of both differential and integral calculus with a minimum of the technique of differentiation and integration.

Beginning with Chapter VI, differential and integral calculus are not kept separate. The derivatives of $\log x$ and $\sin x$ which occur for the first time in this chapter, are not obtained in the usual way. The derivative of $\log_{10} x$ is found before $\log_e x$ by a method which lays stress on the very meaning of logarithms and is as rigorous as the traditional proof. The derivative of $\sin x$ is found by using the properties of circular motion. The logarithmic derivative, which is usually a mere device for finding the actual derivative, is interpreted as the relative rate of change of a quantity y with respect to a quantity x. This is another example of the author's desire to have nothing abstract. Another instance of the same tendency is the definition of curvature as the instantaneous rate of change of α per unit length of arc, where α is the angle made by the tangent with the x-axis or the polar axis.

Chapter VII deals with the technique of integration. There is a good table of integrals in the back of the book, and the student is expected to use it intelligently, not slavishly. This chapter contains applications of single and multiple integrals, and a brief treatment of improper integrals with precautions to be observed in using them.

The first seven chapters (226 pages) have been developed without the law of the mean and the theorems of Taylor and Maclaurin. The manner in which these are introduced and used is one of the strong features of the Davis Calculus. Chapter VIII is on methods of approximation. It begins with a discussion of the elementary methods of finding equations to represent, at least approximately, curves which are given empirically. Lagrange's interpolation formula, the method of differences, and logarithmic plotting are used. Closely related to this subject is the problem of finding polynomial approximations of functions which are not polynomials. Taylor's Theorem is introduced for this purpose. Taylor's series with the remainder is taken up before there is any mention of infinite series, and a study is made of the error involved in neglecting the remainder and using the resulting polynomial approximation for the function under consideration. The work on infinite series is not extensive. Convergence and divergence are defined and illustrated, and a general test for the convergence of the series furnished by the theorems of Taylor and Maclaurin is developed. There is a good paragraph on precautions to be taken in using infinite series.

Chapter IX deals with partial and total derivatives, with applications to plane and solid geometry and to physics. Chapter X is an introduction to ordinary differential equations of the first order. The book closes with 58 pages of useful tables which will relieve the student of many difficulties if he uses them intelligently.

An answer book, containing answers to all of the numerical problems in the book, and indicating in some detail the methods of attack for the theoretical problems, will be published some time this winter. The "Errata," which will

also be printed on a separate sheet, will be printed as a page of the answer book.

At the end of this review of two new books on the calculus, the two lines from Pope, quoted at the beginning of the Davis Calculus, seem appropriate.

"Be not the first by whom the new is tried,
Nor yet the last to lay the old aside."

PROBLEMS AND QUESTIONS.

Under this heading the MONTHLY will act as a medium for the interchange of ideas on behalf of its readers. While questions proposed merely as puzzles will not necessarily be barred by the editors, yet it is hoped that light on many real difficulties may be sought through these columns; and that many problems arising in the actual processes of investigation may be proposed for solution. In order to encourage this latter form of questions and problems, the names of proposers will not be given when the editors are requested to withold them.

A committee of the Editorial Board, consisting of Professor B. F. Finkel, chairman, Drury College, Springfield, Mo.; Professor R. P. Baker, University of Iowa, and Professor R. D. Carmichael, Indiana University, will have charge of contributions to this department. Correspondence may be addressed to the chairman or to any member of the committee. No problems are printed this month owing to the unusual space occupied by other matter. This department will be resumed in the February issue.

NEWS AND NOTES.

Under this heading the MONTHLY will print information concerning new appointments and promotions of teachers of mathematics, notices of meetings of mathematical associations, biographical and bibliographical notes, references to articles of interest in other mathematical journals, courses in mathematics offered in American universities and any items whatsoever of general interest to teachers of mathematics.

Contributions to this column should be sent to Professor FLORIAN CAJORI, Colorado College, Colorado Springs, Colo., who is chairman of the committee having this department in charge. Other members are Professor G. A. MILLER, University of Illinois, Professor W. H. BUSSEY, University of Minnesota, and Professor W. DE W. CAIRNS, Oberlin College.

Mr. E. R. SMITH, formerly of the Brooklyn Polytechnic Institute, is now head master of the Park School, Baltimore, Md.

Mr. GUY H. ALBRIGHT, assistant professor of mathematics at Colorado College will spend the second semester of the present academic year at Harvard University and will give his time mainly to the pursuit of graduate work.

Professor G. A. MILLER, of the University of Illinois, has been elected corresponding member of the Spanish Mathematical Society.

Professor DAVID EUGENE SMITH, of Columbia University, delivered an address on "The Teaching of Mathematics" at the annual meeting of the Kansas State Teachers' Association held at Topeka in November, 1912.

Professor H. E. SLAUGHT, of the University of Chicago, delivered an address on "The Present Trend in the Teaching of Secondary Mathematics" at the Iowa State Teachers' Association held at Des Moines in November, 1912.

At the meeting of the college section of the Illinois State Teachers' Association, held at Peoria December 27, 1912, "The Preservation of the Four Years' College Course" was a topic for discussion led by President THOMAS MCCLELLAND of Knox College.

The winter meeting of the Chicago Section of the American Mathematical Society was merged in the annual meeting of the Society, held at Cleveland, Ohio, December 31, 1912, to January 2, 1913. An account of this meeting will be given in the February issue of the MONTHLY.

Mr. WILLIAM BETZ, head of the department of mathematics in the East High School, Rochester, N. Y., addressed the meeting of the Principals' Association of New York State at Rochester, December 27, 1912, on the "Final Report of the National Committee of Fifteen on Geometry Syllabus."

The *Mathematics Teacher* for December, 1912, contained the report of the National Committee of Fifteen on Geometry Syllabus in its final form, as presented and accepted at the Chicago meeting of the National Education Association in July, 1912. A large edition of reprints of this report is now ready for distribution through the Commissioner of Education, Washington, D. C. These may be had *gratis* upon application.

The Southern Association of Colleges and Preparatory Schools held its annual meeting at Spartanburg, S. C., in November, 1912. Membership in this association is based upon the standards set by the Carnegie Foundation for the advancement of teaching as applied to both colleges and secondary schools. The only college added to the membership at this meeting was Converse College for women at Spartanburg, where Professor T. MCNIDER SIMPSON is the aggressive head of the department of mathematics.

The following letter from Professor B. F. YANNEY, of the University of Wooster, Ohio, speaks for itself: "I note with interest and pleasure the changes proposed in respect to the AMERICAN MATHEMATICAL MONTHLY. The new control makes secure its future career. The wider scope of its aims insures still greater usefulness to its patrons. It has been an important source of help to me during the entire period of its existence thus far, and now it promises to be even more so in the years to come. It deserves a steadily growing success, which is my wish and prediction for it."

Mr. MIKAMI shows in the *Archiv der Mathematik und Physik*, volume XX, page 1, that the proof of the Pythagorean theorem usually attributed to the Hindu Bhaskara (about 1150 A. D.)—based on a figure in which a right triangle is drawn four times in the square on the hypotenuse, so that in the middle there remains a square whose side equals the difference between the two legs of the right triangle—was given early in the Christian era by the Chinese Chang Chun-Ch'ing.

Under the general title "Present Civilization" (Kultur der Gegenwart), the well-known German publisher, B. G. TEUBNER of Leipzig has announced a long series consisting of sixty volumes. Eighteen of these are to be devoted to the mathematical and the natural sciences. The part devoted to the mathematical sciences is already in press. It is edited by F. Klein, with the collaboration of P. Stäckel, H. E. Timerding, A. Voss, and G. H. Zeuthen.

A letter from B. G. Teubner dated November 23, 1912, states that the remaining two parts of Pascal's "Repertorium der höheren Mathematik" can not appear within the next two or three months. Recent circulars issued by the publishers stated that this work would appear at the end of 1912. The new edition consists of four volumes, and gives in a convenient form a statement of the fundamental theorems of pure mathematics. A similar work devoted to applied mathematics, entitled "Repertorium der Angewandte Mathematik" is also to appear in four volumes, the first of which is expected in the spring of 1913.

The October number of the journal of the new Spanish Mathematical Society, entitled "*Revista de la Sociedad Matematica Espanola*," calls attention to the fact that there were twenty-five Spanish mathematicians at the International Mathematical Congress held at Cambridge, England, last August. The numbers of the Spanish mathematicians attending the four preceding international mathematical congresses were 1, 3, 1, and 5 respectively. It would appear that the mathematical interest in Spain is rapidly growing at the present time, and it is to be hoped that this will continue to grow under the influence of the new national mathematical society.

A new weekly scientific journal called the *Natur Wissenschaften* is to be started in Germany in January, 1913. This journal is expected to occupy the field covered by *Nature* in England, and *Science* in America. The technical applications of the sciences are, however, to be considered more fully than in these English and American journals, and mathematics is to receive as much attention as may be possible in such a general journal.

The annual meeting of the Southwestern Section of the American Mathematical Society was held at Lincoln, Neb., November 30, 1912. Papers were presented by Professor W. H. Roever of Washington University, Professor S. W. Reaves of the University of Oklahoma, Dr. E. R. Bennett of the University of Nebraska (two papers), Professor Arnold Emch of the University of Illinois, Professor R. D. Carmichael of Indiana University, Professor Florian Cajori of Colorado College, Dr. S. Lefschetz of the University of Nebraska, Dr. E. L. Dodd

of the University of Texas, Dr. T. H. Gronwall, Chicago, Ill. (four papers), Professor E. R. Hedrick of the University of Missouri, and Professor Louis Ingold of the University of Missouri. Abstracts of these papers will be printed in the *Bulletin* of the American Mathematical Society.

Of the 177 publications of the Carnegie Institution at Washington, as shown in the latest catalog, five are in pure mathematics as follows: "Synopsis of Linear Associative Algebra" by James Byrnie Shaw, University of Illinois; "Factor Table for the First Ten Millions" and also "Tables Giving a Complete List of Prime Numbers between the Limits 1 and 10,006,721," by Derrick N. Lehmer, University of California; "A Sylow Factor Table of the First Twelve Thousand Numbers" by Henry W. Stager, University of California; and "The Symmetric Function Tables of the Fifteenthic" by Floyd F. Decker, Syracuse University. There are also some other papers of a mathematical nature including one by F. R. Moulton of the University of Chicago on "Periodic Orbits" and several in collaboration with Professor T. C. Chamberlin on "Contributions to Cosmogony and the Fundamental Problems of Geology." The latter are by William D. MacMillan, Arthur C. Lunn, and F. R. Moulton of the University of Chicago, and Charles S. Slichter of the University of Wisconsin.

An outline of the report presented by A. Gutzmer at the Mathematical Congress in Cambridge, England, on the work done by the German sub-committee of the International Commission on the Teaching of Mathematics, is printed in *Science* for December 13, 1912. The German report is composed of five volumes: (1) The secondary schools of northern Germany; (2) The secondary schools of southern and middle Germany. (3) Special problems of secondary mathematics instruction; (4) Mathematics in the technical schools; (5) The teaching of mathematics in elementary schools, and in the training schools for elementary teachers. A significant paragraph in this German report is the following: "This reorganization of secondary education aims at making the youth of our country sympathetic with labor as well as appreciative of the best in modern culture. From this point of view the teaching of mathematics and science assumes a position equivalent to that in history and languages."

It is just announced, as we go to press, that Doctor Oskar Bolza, Honorary Professor of Mathematics, University of Freiburg, Freiburg, Germany, will lecture at the University of Chicago during the summer session of 1913. The topics upon which he will lecture are: Linear Integral Equations and Theory of Functions of a Complex Variable. This is his first return since his retirement from Chicago as Non-Resident Professor. His many friends in this country will welcome him most heartily.

2

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HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

By FflorIAN CAJORI, Colorado College.

THE MODERN EXPONENTIAL NOTATION

(Continued).

J. H. Rahn's *Deutsche Algebra*, printed in Zürich, 1659, contains for positive integral powers two notations, one using the cartesian exponents, a^3, x^4, the other consisting in writing a small spiral between the base and the exponent on the right. Thus $a \supset 3$ signifies a^3. The spiral signifies *involution*, a process which he calls *involviren*. An English translation, altered and augmented by John Pell, was made by Thomas Brancker and published 1668 in London.[1] In the same year positive integral exponents were used by Lord Brouncker in an early volume of the *Philosophical Transactions* of London.[2] In these transactions none of the pre-descartian notations for powers appear, except a few times in an article of 1714, written by John Cotes.

Of interest is the following passage in Newton's *Universal Arithmetick* (which consists of lectures delivered at Cambridge in the period, 1669–1685 and first printed 1707): "Thus $\sqrt{64}$ denotes 8; and $\sqrt{3:64}$ denotes 4. . . . There are some, that to denote the Square or first Power, make use of q, and of c for the Cube, qq for the Biquadrate, and qc for the Quadrato-Cube, etc. . . . Others make use of other sorts of Notes, but they are now almost out of Fashion."[3] In an edition of 1679 of the works of Fermat the algebraic notation of Vieta, originally followed by Fermat, is discarded in favor of the exponents of Descartes.[4] It would seem, from what has been cited, that about 1660 or 1670 the positive integral exponent had won an undisputed place in algebraic notation. Though generally adopted, it was not universally so. A large volume, the P. Gasparis Schotti *Cursus mathematicus*, Frankfurt a. M., 1661, and the second edition of Diophantus by Bachet de Méziriac (Toulouse, 1670), contain no trace of the modern exponential notation. Joseph Raphson's explanation of his method

[1] See G. Wertheim in *Bibliotheca mathematica*, 3d S., Vol. III, 1902, pp. 113–126.

[2] *Phil. Trans.*, Vol. III, for anno 1668, printed 1669, p. 647.

[3] Newton's *Universal Arithmetick*, London, 1728, p. 7.

[4] *Œuvres de Fermat.* Éd. PAUL TANNERY ET CHARLES HENRY, T. I, Paris, 1891, p. 91 foot-note.